

# Analytic evaluation of Feynman graph integrals \*

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We review the main steps of the differential equation approach to the analytic evaluation of Feynman graphs, showing at the same time its application to the 3-loop sunrise graph in a particular kinematical configuration.

- The differential equation approach to the analytic evaluation of Feynman graph integrals applies to loop integrals defined in the by now customary regularization in  $d$  continuous dimensions. We will recall the main steps of the approach, describing as an example its application to the 3-loop sunrise graph in the kinematical configuration of Fig. 1.

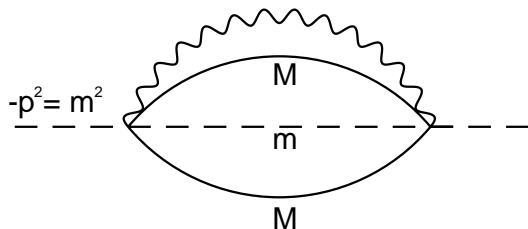


Fig. 1. The considered 3-loop sunrise graph.

- As the very first step, for any given graph one has to take all the scalar products which can be formed with the external momenta and all the loop momenta. In the case of Fig. 1 there is a single external momentum  $p$  and three loop momenta  $k_i$ ,  $i = 1, 2, 3$ , so that, irrespectively of the actual structure of the graph, the scalar products are the three  $(p \cdot k_i)$ , the three  $k_i^2$  and the three  $(k_i \cdot k_j)$  for  $i \neq j$ , for a total of nine. One can

choose to express some of the scalar products as linear combination of the denominators (in general, there are several possible choices, equivalent for the following); the surviving scalar products are said irreducible (perhaps better not trivially reducible). One then considers the whole family of all the integrals obtained by taking as integrand all the possible combinations of the  $P$  denominators of the scalar propagators raised to integer powers  $\alpha_q$ ,  $q = 1, \dots, P$  and of the  $S$  irreducible scalar products raised to integer powers  $\beta_r$ ,  $r = 1, \dots, S$ , the continuous dimensional regularization ensuring the absence of convergence problems. The corresponding integrals are functions of the continuous dimension  $d$ , of the Mandelstam variable  $p^2$  and of the set of  $(P + S)$  indices  $\alpha_q, \beta_r$ . (For Fig. 1,  $P = 4$  and  $S = 5$ ). For the following discussion, let us further define  $A = \sum_q (\alpha_q - 1)$ , with  $\alpha_q \geq 1$  and  $B = \sum_r \beta_r$ , with  $\beta_r \geq 0$ .

- As next step, we look for relations between the previous integrals. The relations are best provided by means of the by now famous integration by parts identities (*ibp-id's*) [1], obtained by replacing in any of the above integrals the integrand by all the combinations of the kind  $(\partial/\partial k_{i,\mu})v_\mu$ , times the same integrand, where  $i = 1, \dots, 3$  and  $v_\mu$  is any of the external or loop momenta. (For the graph of Fig. 1 there are  $3 \times 4 = 12$  different combinations  $(\partial/\partial k_{i,\mu})v_\mu$ .)

By carrying out explicitly the derivatives, and after some trivial algebra, one is left with a combi-

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nation of polynomials (depending in general on  $d$ , on the Mandelstam variables, and on the internal masses, always with integer coefficients) times integrals of the same family. But the whole original integral vanishes, as the integrand is by construction a divergence, so that the combination of integrals vanishes as well – an integration by parts identity has been obtained. If the original integrand belongs to the class  $(A, B)$ , *i.e.* if the sums of the powers of denominators and numerators are  $(A - P, B)$ , the involved integrals belong to the classes  $(A \pm 1, B \pm 1)$ . Integrals in which one of the propagator is missing (*i.e.* its exponent is 0) are said to correspond to subtopologies, and can be considered as already known in a systematic bottom up approach. (It is to be observed that other identities of similar nature can be obtained by exploiting the Lorentz Invariance [2] on the external vectors, but only when there are at least three independent external vectors.)

- At this point, it is natural to try to exploit the identities for expressing as many as possible of the integrals actually needed in any explicit calculation in terms of as few as possible integrals of the set – the so-called master integrals. The original ideas for building algorithms applying to the most general case goes back to Laporta ([2]-[5]): the integrals are ordered by giving them a *weight* (that can be done almost at will by giving a higher weight to the integrals considered more complicated; typically, the fully scalar integral is given the lowest weight), and then a “sufficient number” of explicit equations is written for them. In the case of Fig. 1, starting from the fully scalar integral, corresponding to  $(A = 0, B = 0)$ , one obtains 12 *ibp-id*’s involving, besides the original integral, the  $4+5+20 = 29$  integrals with  $(A = 1, B = 0)$ ,  $(A = 0, B = 1)$  and  $(A = 1, B = 1)$  – an apparently runaway situation, in which the involved integrals outnumber the equations. But that is not the case; an elementary combinatorial calculation shows that by writing all the *ibp-id*’s starting from  $A = 0, 1$  and  $B = 0, 1$  one generates  $12 \times 30 = 360$  equations, involving integrals with  $A = 0, 1, 2$  and  $B = 0, 1, 2$  at most; as the number of all the integrals with  $A, B = 0, 1, 2$  is just 336 one has more equations than integrals, *i.e.* an apparently over-

constrained system. That is not the case either – simply, the equations are not all linearly independent.

The system can be solved by Gauss substitution, eliminating first the integrals of higher weight – the method is elementary, even if, given the size of the system, it is convenient to use an automatized computer procedure. When that is done, it is found that all the integrals appearing when dealing with the graph of Fig.1. can be expressed in terms of 4 Master Integrals (MI’s).

- Let  $F(d, p^2)$  be any of the MI’s. Obviously,

$$p^2 \frac{\partial}{\partial p^2} F(d, p^2) = 2p_\mu \frac{\partial}{\partial p_\mu} F(d, p^2).$$

Substitute in the *r.h.s.* for  $F(d, p^2)$  its original definition as integral on the loop momenta of a suitable integrand. By carrying out explicitly the derivative  $\partial/\partial p_\mu$  and then contracting with  $p_\mu$  one obtains a combination of integrals associated to the considered graphs, which can then be expressed, thanks to the solution of the *ibp-id*’s discussed in the previous paragraph, as combination of the MI’s. From the above equation one has therefore expressed the  $p^2$ -derivative of the particular MI  $F(d, p^2)$  in terms of the MI’s of the problem. By repeating the procedure for all the MI’s, one obtains a system of linear first order differential equations for the MI’s. The system is in general non homogeneous, as the *ibp-id*’s can introduce integrals where some of the propagators are missing. The non-homogenous part involves simpler graphs, and can therefore be considered as known in a systematic bottom up approach.

If the MI’s depend on several independent momenta  $p_\mu^i$  and correspondingly by several Mandelstam variables, by acting on the MI’s with all the combinations  $p_\mu^i \partial/\partial p_\mu^j$  and minor algebraic rearrangements one can obtain a system of differential equations in any of the Mandelstam variables, [6]. It is to be recalled here that the solutions of the *ibp-id*’s can be almost immediately rewritten as differential equations in the internal masses [7]. Finally, a linear system of differential equations can always be rewritten as a single differential equation of suitable higher order for any of its unknown functions. The coefficients appearing in the equations are in any case polynomials (or rational factors, when dividing the equations by the polynomial multiplying the highest derivative).

• In the case of the special kinematical situation depicted in Fig. 1, one of the internal masses vanishes, two internal masses take the same value  $M$ , the fourth the value  $m$  and the Mandelstam variable is timelike and takes the value  $m^2$ , *i.e.*  $p^2 = -m^2$ . We rescale all the momenta by  $M$ , further put  $m = Mx$  and define

$$\Phi(d, x) = \frac{C^{-3}(d)}{(2\pi)^{3(d-2)}} \int d^d k_1 d^d k_2 d^d k_3 \times \frac{1}{k_1^2(k_2^2 + 1)(k_3^2 + 1)[(p - k_1 - k_2 - k_3)^2 + x^2]}, \quad (1)$$

where  $C(d) = (4\pi)^{2-d/2} \Gamma(3 - d/2)$  is a normalization factor, introduced for convenience, with the limiting value  $C(4) = 1$  at  $d = 4$ . The above discussed equations in  $p^2$  and the masses give for  $\Phi(d, x)$  the following 3rd O.D.E., exact in  $d$ :

$$\left[ \sum_{j=1}^3 \mathcal{P}_j(d, x) \frac{d^j}{dx^j} + \mathcal{P}_0(d, x) \right] \Phi(d, x) = \mathcal{N}(d, x), \quad (2)$$

where

$$\begin{aligned} \mathcal{P}_3(d, x) &= x^2(1-x)(1+x); \\ \mathcal{P}_2(d, x) &= [2 - (d-4)]x + [2 + 5(d-4)]x^3; \\ \mathcal{P}_1(d, x) &= -[6 + 8(d-4) + 2(d-4)^2] \\ &\quad + [2 - 4(d-4) - 6(d-4)^2]x^2; \\ \mathcal{P}_0(d, x) &= -[8 + 14(d-4) + 6(d-4)^2]x; \\ \mathcal{N}(d, x) &= \frac{1}{(d-4)^3} \frac{x^{(d-2)}}{x}. \end{aligned} \quad (3)$$

• The direct solution of the differential equation, for arbitrary  $d$ , seems out of reach. But as, in any case, one is interested in the  $d \rightarrow 4$  limit of the solution, one can expand  $\Phi(d, x)$  in Laurent series in  $(d-4)$  and rewrite the equation in terms of the coefficients of the expansion. Let us take as known, for simplicity, that in our case the leading (most singular) term is of order  $1/(d-4)^3$ , so that the expansion of  $\Phi(d, x)$  reads

$$\Phi(d, x) = \sum_{k=-3} (d-4)^k \Phi^{(k)}(x). \quad (4)$$

By correspondingly expanding Eq.(2) in  $(d-4)$ , one obtains a system of chained equations for the  $\Phi^{(k)}(x)$ ; the first equation involves only  $\Phi^{(-3)}(x)$ ,

and is to be solved for  $\Phi^{(-3)}(x)$  independently of the other coefficients; the second equation is an equation for  $\Phi^{(-2)}(x)$  involving  $\Phi^{(-3)}(x)$  in the non-homogeneous term, and so on. The general equation reads

$$\begin{aligned} & \left[ \sum_{j=1}^3 P_j(x) \frac{d^j}{dx^j} + P_0(x) \right] \Phi^{(k)}(x) \\ &= \left[ \sum_{l=1}^2 Q_l(x) \frac{d^l}{dx^l} + Q_0(x) \right] \Phi^{(k-1)}(x) \\ &+ \left[ R_1(x) \frac{d}{dx} + R_0(x) \right] \Phi^{(k-2)}(x) \\ &+ S_0(x) \frac{1}{(k+3)!} \ln^{(k+3)}(x) \end{aligned} \quad (5)$$

where  $\Phi^{(k)}(x) = 0$  if  $k < -3$ , and  $P_i(x)$ ,  $Q_i(x)$ ,  $R_i(x)$ ,  $S_0(x)$  are simple polynomials in  $x$  whose explicit expressions can be easily derived from Eq.(2).

All the above equations have the same form

$$\mathcal{D}(x) \Phi^{(k)}(x) = N^{(k)}(x), \quad (6)$$

where  $\mathcal{D}(x)$  is the differential operator applied to  $\Phi^{(k)}(x)$  in the *l.h.s.* of Eq.(5) and  $N^{(k)}(x)$  stands for the non-homogeneous part, to be considered as known when dealing with the order  $k$  in the  $(d-4)$  expansion. The homogeneous part, which is the same for any  $k$ , can be written as

$$\begin{aligned} \mathcal{D}(x) \Phi^{(k)}(x) &\equiv \left[ \frac{d^3}{dx^3} + 2 \left( \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} \right) \frac{d^2}{dx^2} \right. \\ &- 2 \left( \frac{3}{x^2} + \frac{1}{1-x} + \frac{1}{1+x} \right) \frac{d}{dx} \\ &\left. - 2 \left( \frac{3}{x^2} + \frac{1}{1-x} + \frac{1}{1+x} \right) \right] \phi(x) \\ &= 0. \end{aligned} \quad (7)$$

• We will try to solve the equations for the coefficients of the  $(d-4)$  expansion by means of the method of the variation of constants by Euler. To that aim, we need the solutions of the homogeneous equation, Eq.(7).

Up to here, our approach applies to virtually any Feynman graph integral; for continuing one has to solve the homogeneous differential equation in

$d = 4$  dimensions. Algorithms for the solution of differential equations in the general case are unfortunately not available. But as a more optimistic remark, let us point out that in almost all the cases considered sofar (see for instance [8], [9], [10]) the homogeneous equations came out to be rather simple, with solutions which could be expressed in terms of almost elementary functions. The same happens also in the present case. Indeed, it is immediately seen that a first solution of Eq.(7) is

$$\phi_1(x) = (1 - x^2) ;$$

by putting  $\phi(x) = \phi_1(x)\psi(x)$  one finds a second order equation for  $\psi'(x)$  having the simple solution

$$\psi'_1(x) = (1 - x^2)^2/x^3 ,$$

and finally, putting  $\psi'(x) = \psi'_1(x)\chi(x)$ , one finds at once

$$\chi'(x) = -x^4/(1 - x^2)^5 .$$

The corresponding solutions are

$$\begin{aligned} \phi_2(x) &= \\ &- \frac{1}{2} \frac{(1 - x^2)(1 - x^4)}{x^2} - 2(1 - x^2)H(0; x) , \end{aligned}$$

$$\begin{aligned} \phi_3(x) &= \\ &+ \frac{3}{512} \frac{(1 - x^2)(1 - x^4)}{x^2} [H(-1; x) + H(1; x)] \\ &+ \frac{3}{128} (1 - x^2) [H(0, -1; x) + H(0, 1; x)] \\ &- \frac{1}{256} \frac{(x^2 + 3)(3x^2 + 1)}{x} , \end{aligned}$$

where the functions  $H$  are Harmonic Polylogarithms or HPLs [11]. Referring to [11] for more details, let us recall that the HPLs depends on an argument, say  $x$ , and on a vector of indices, say  $\vec{b}$ , whose components take any of the values  $(1, 0, -1)$ , and whose number is called the weight ( $w$ ). At  $w = 1$ , they are defined as

$$\begin{aligned} H(0, x) &= \ln(x) , \\ H(1, x) &= -\ln(1 - x) = \int_0^x \frac{dx'}{1 - x'} , \\ H(-1, x) &= \ln(1 + x) = \int_0^x \frac{dx'}{1 + x'} ; \end{aligned}$$

at higher weight,  $w > 1$ , if the vector of the indices is written as  $(a, \vec{b})$ , where  $\vec{b}$  is a vector of  $w - 1$  components, they fulfill

$$\frac{d}{dx} H(a, \vec{b}; x) = f(a, x) H(\vec{b}; x) , \quad (8)$$

with

$$\begin{aligned} f(1, x) &= 1/(1 - x) , \\ f(0, x) &= 1/x , \\ f(-1, x) &= 1/(1 + x) . \end{aligned} \quad (9)$$

The Wronskian of the  $\phi_i(x)$ , as expected by direct inspection of Eq.(7), is

$$W(x) = -(1 - x)^2(1 + x)^2/x^2 .$$

In terms of the solutions of the homogeneous equation, the general solution of the inhomogeneous equation is given by

$$\begin{aligned} \Phi^{(k)}(x) &= \\ \phi_a(x) &\left[ \Phi_a^{(k)} + \int^x dy \frac{\epsilon_{abc} M_{bc}(y)}{W(y)} N^{(k)}(y) \right] , \\ M_{bc}(y) &= \phi_b(y) \phi'_c(y) - \phi'_b(y) \phi_c(y) , \end{aligned} \quad (10)$$

where  $\{a, b, c\} = p\{1, 2, 3\}$  and  $\Phi_a^{(k)}$  ( $a = 1, 2, 3$ ) are three integration constants to be fixed, order by order in  $k$ , to match the boundary conditions identifying  $\Phi(d, x)$  Eq.(1).

• Let us look at Eq.(10) for the first value of  $k$ , which is  $k = -3$ . From Eqs. (5) one gets

$$N^{(-3)}(x) = \frac{1}{2} \left( \frac{2}{x} + \frac{1}{1 - x} - \frac{1}{1 + x} \right) ;$$

when it is inserted in Eq.(10), after partial fractioning and some trivial integration by parts, the remaining non trivial integrals involve only one of the three factors  $1/x, 1/(1 - x), 1/(1 + x)$ , the three factors  $f(a, x)$  of Eq.(9) and polylogarithms of various weight: but those (indefinite) integrals can be carried out at once by using Eq.(8). It follows that  $N^{(-2)}(x)$ , which contains  $\Phi^{(-3)}(x)$ , is also a combination of powers of the monomials  $x, (1 - x), (1 + x)$  and polylogarithms, so that the procedure can be iterated at will up to any desired value of  $k$ , the result being, at all orders, a combination of harmonic polylogarithms.

• To complete the job, we must fix the so far arbitrary constants  $\Phi_a^{(k)}$ ,  $a = 1, 2, 3$ . To that aim, let us observe that the behaviour of the most general solution of Eq.(2) is of the kind

$$\Phi(d, x) = \sum_{i=1}^4 C_i x^{\alpha_i} (1 + c_{i,1} x^2 + c_{i,2} x^4 + \dots) ,$$

where the four possible values of the exponents are  $\alpha_1 = 0$ ,  $\alpha_2 = -(d-2)$ ,  $\alpha_3 = (2d-5)$ , and  $\alpha_4 = (d-2)$ , as can be easily verified by substituting the above expansion in Eq.(2). On the other hand, by direct inspection of  $\Phi(d, x)$  Eq.(1), one finds that the behaviours  $\alpha_2$  and  $\alpha_3$  are ruled out, as the integral does not diverge in  $x$  in the  $x \rightarrow 0$  limit when  $d > 2$ ; that implies that terms like  $1/x^2$  or  $1/x$  (or more in general terms with odd powers of  $x$ ) cannot be present in the expansion around  $x = 0$  in the  $d \rightarrow 4$  limit. Similarly, one finds by inspection that  $\Phi(d, x)$  is analytic in  $x$  for  $x \rightarrow 1$ ; that implies that terms in  $\ln(1-x)$  cannot be present in the expansion around  $x = 1$  in the  $d \rightarrow 4$  limit. Those conditions are sufficient to fully determine the integration constants. Indeed, we evaluated explicitly Eq.(10) up to  $k = 5$ , and by imposing the proper behaviours for  $x \rightarrow 0$ ,  $x \rightarrow 1$  we fixed the constants  $\Phi_1^{(k)}$ ,  $\Phi_2^{(k)}$ ,  $\Phi_3^{(k)}$  for  $-3 \leq k \leq 3$ , ( $\Phi_3^{(k)}$  being 0 for any  $k$ ) and  $\Phi_2^{(4)}$ . The constants  $\Phi_1^{(4)}$ ,  $\Phi_1^{(5)}$  and  $\Phi_2^{(5)}$  remained still undetermined - to fix them, one has to impose the boundary conditions to further terms  $\Phi^{(k)}(x)$  with  $k = 6, 7$ .

The explicit analytic solution up to  $k = 3$  included, which we obtained in that way, involves HPL's of argument  $x$  and up to weight  $w = 6$ . The resulting expression is unfortunately too long to be listed here. More details can be found in [10].

• Having the full dependence on  $x$  in closed analytic form it is immediate to obtain the values at  $x = 0$  and  $x = 1$ . The  $x = 0$  values can also be obtained by direct integration of Eq.(1); they can be used as a check of the calculation.

The values at  $x = 1$  are more interesting. They correspond to the quantity  $J_{11}$  of [3], where they were given up to order  $(d-4)^3$ ; from our solution (and from the table [12] of the values of the HPL's at  $x = 1$ ), we could evaluate them up to order

$(d-4)^5$  included; in fact, given the structure of the solutions of the homogeneous equation, the still undetermined constants are not needed for the  $x = 1$  value. The results up to  $(d-4)^3$  are in full agreement with [3]; the new terms, namely the coefficients of order  $(d-4)^4$  and  $(d-4)^5$  match the numerical values of the quantity  $I_{12}$  of [13].

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